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Journal of Computational and Applied Mathematics 166 (2004) 31–54
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Uniform convergence difference schemes for singularly perturbed mixed boundary problems☆

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Received 20 October 2002; received in revised form 3 May 2003

Abstract

In this paper, we consider the conservative form of singularly perturbed ordinary differential equations with mixed boundary conditions. A fitted mesh finite difference scheme is constructed for these problems. The scheme is shown to be uniformly convergent with respect to the perturbed parameter. A class of conservative difference schemes with uniform mesh are also considered. These difference schemes are proved to be first-order uniformly convergent. The computed results for both cases are in good agreement with the exact solutions.

PACS: 45.10.Hj; 44.05.+e; 02.60.−x; 02.70.Bf

Keywords: Mixed boundary problems; Singular perturbation; Fitted mesh difference scheme; Conservative difference scheme; Uniform convergence

1. Introduction

Mathematical models of physical or chemical problems, in which some material parameters are small, can be successfully studied, in many cases, as singularly perturbed problems [3,4,7–14,16]. Therefore, the interest in developing and analyzing efficient numerical methods for singularly perturbed problems has increased enormously.

☆ This work has been supported by the National Natural Science Foundation of China under Grant 10271098.
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0377-0427/$ - see front matter
doi:10.1016/j.cam.2003.09.038

In this paper, we consider the following conservative form of singularly perturbed mixed boundary problems (SPMBP):

$$Lu(x) \equiv \varepsilon(p(x)u'(x))' + (q(x)u(x))' + r(x)u(x) = f(x), \quad x \in \Omega, \tag{1.1}$$

$$B_0u(0) \equiv \alpha^* u(0) - \beta^* u'(0) = A, \tag{1.2}$$

$$B_1u(1) \equiv \gamma^* u(1) + \delta^* u'(1) = B, \tag{1.3}$$

where ε $(0 < \varepsilon \ll 1)$ is a sufficiently small parameter, $\Omega = (0,1)$, $\bar{\Omega} = [0,1]$, $p(x), q(x), r(x), f(x)$ are sufficiently smooth in $\bar{\Omega}$ and satisfy

$$\bar{\alpha} > p(x) > \underline{\alpha} > 0, \ \bar{\beta} > q(x) > \underline{\beta} > 0, \ r(x) \leqslant 0,$$

$$\bar{p} > p'(x) \geqslant 0, \ q'(x) \leqslant 0,$$

$$\alpha^*, \beta^*, \gamma^*, \delta^* > 0.$$

It is well known that the solution of the SPMBP converges, as $\varepsilon \to 0$, and for $0 < x \leqslant 1$, to the solution of the reduced problem. The loss of a boundary condition at $x = 0$ in the reduced problem results in a boundary layer in the solution u, for a small ε.

Computational methods for the conservative form of singular perturbed ordinary differential equations have been studied in different ways [1,15]. In this paper, two kinds of schemes are constructed for these problems with mixed boundary conditions. One is a fitted mesh finite difference scheme (FMFDS), another kind is a class of conservative difference schemes (CCDS) with uniform mesh. Both schemes are very interesting and useful schemes.

To obtain the required estimates of SPMBP is more difficult. In the traditional argument of FMFDS, the solution u is usually decomposed into smooth component v and singular component w. However, the singular component w may cause very “large” at boundary layer $x = 0$, because the boundary layer at $x = 0$ contains a term $u'(0)$.

In this paper, we use Kellogg and Tsan’s [5] methods to decompose solution u. Then we combine Kellogg and Tsan’s techniques and Shishkin’s techniques [13] to prove FMFDS to be uniformly convergent. We use Kellogg and Tsan’s techniques to prove that these conservative difference schemes to be first-order uniformly convergent. Numerical results are compared with the exact solutions. We also show that both schemes give good agreement with the exact solutions.

2. Properties of the solution of SPMBP

For convenience, let

$$d(x,\varepsilon) = \frac{\varepsilon \cdot p'(x) + q(x)}{p(x)}, \quad d = \min_{x \in [0,1]} \frac{q(x)}{2p(x)} > 0,$$

$$\bar{d} = \max_{x \in [0,1]} \frac{p'(x) + q(x)}{p(x)} > 0$$

then

$$\bar{d} \geqslant d(x,\varepsilon) \geqslant 2d > 0.$$

We have given some properties of solution of SPMBP in [2]. The main result can be written in the following lemma.

Lemma 1. *Suppose $u(x)$ is the solution of SPMBP, then*

$$u(x) = e \cdot v(x) + z(x) \tag{2.1}$$

where

$$v(x) = \exp\left(-\frac{q(0)\cdot x}{p(0)\cdot \varepsilon}\right), \tag{2.2}$$

$$|e| \leqslant C\varepsilon \tag{2.3}$$

and $z(x)$ satisfies

$$|z^{(i)}(x)| \leqslant C \cdot \left\{1 + \varepsilon^{-i+1}\exp\left(-\frac{d\cdot x}{\varepsilon}\right)\right\} \quad (i = 0, 1, \ldots). \tag{2.4}$$

Throughout the paper, we let $C, C_0, C_1, \ldots$ and K_1, K_2, K_3 denote positive constants that may take different values in different formulas, but that are always independent of h and ε.

3. A fitted mesh finite difference scheme

In this section, a fitted mesh finite difference scheme for SPMBP is now constructed. On Ω, a piecewise-uniform mesh of N mesh intervals is introduced with the transition point $\sigma = \min\{\frac{1}{2}, \varepsilon \ln N/d\}$. The domain $\bar{\Omega}$ is subdivided into two subintervals: $[0, \sigma]$ and $[\sigma, 1]$. On each subinterval a uniform mesh with $N/2$ mesh points is placed.

Let $h_l = 2\sigma/N$; $h_r = 2(1-\sigma)/N$; $h_i = h_l$, when $1 \leqslant i \leqslant N/2$; $h_i = h_r$, when $N/2 < i \leqslant N$; $\bar{h}_i = (h_i + h_{i+1})/2$, $\bar{h} = (h_l + h_r)/2$; $x_0 = 0$ and $x_i = x_{i-1} + h_i$, $(i = 1, 2, \ldots, N)$.

A FMFDS is defined by

$$L^N u_i \equiv \varepsilon \cdot \delta(p(x_i)\cdot \delta u_i) + D_+(q(x_i)u_i) + r(x_i)u_i = f(x_i), \quad 0 < i < N, \tag{3.1}$$

$$B_0^N u_0 \equiv \alpha^* u_0 - \beta^* \frac{u_1 - u_0}{h_l} = A, \tag{3.2}$$

$$B_1^N u_N \equiv \gamma^* u_N + \delta^* \frac{u_N - u_{N-1}}{h_r} = B, \tag{3.3}$$

where

$$\delta u_i = \frac{u_{i+1/2} - u_{i-1/2}}{\bar{h}_i}, \quad \delta u_{i+1/2} = \frac{u_{i+1} - u_i}{h_{i+1}},$$

$$\delta u_{i-1/2} = \frac{u_i - u_{i-1}}{h_i}, \quad D_+(q(x_i)u_i) = \frac{q(x_{i+1})u_{i+1} - q(x_i)u_i}{h_{i+1}},$$

$$\delta(p(x_i)\cdot \delta u_i) = \frac{p(x_{i+1/2})\delta u_{i+1/2} - p(x_{i-1/2})\delta u_{i-1/2}}{\bar{h}_i}.$$

Lemma 2. *If the mesh function u_i satisfies $L^N u_i \leqslant 0$, $B_0^N u_0 \geqslant 0$, $B_1^N u_N \geqslant 0$, then $u_i \geqslant 0$ for all $0 \leqslant i \leqslant N$.*

Proof. The fitted mesh finite difference scheme (3.1)–(3.3) may be written in matrix vector form

$$A \cdot U = F, \tag{3.4}$$

where

$$F = (-B_0^N u_0, L^N u_1, L^N u_2, \ldots, L^N u_{N-1}, -B_1^N u_N)',$$

$$U = (u_0, u_1, \ldots, u_N)'.$$

In (3.4) A is a tridiagonal matrix and the matrix A is an irreducible M matrix [17]. Thus, A^{-1} exists and $A^{-1} \leqslant 0$. By using hypothesis $F \leqslant 0$, it follows that $U \geqslant 0$. □

Lemma 3. *If the mesh function u_i satisfies*

$$|L^N u_i| \leqslant K_1, \quad 1 \leqslant i \leqslant N-1, \tag{3.5}$$

$$|B_0^N u_0| \leqslant K_2, \tag{3.6}$$

$$|B_1^N u_N| \leqslant K_3, \tag{3.7}$$

then $|u_i| \leqslant C$ for all $0 \leqslant i \leqslant N$.

Proof. Without loss of generality we suppose $\Delta = \gamma^*(\alpha^* + \beta^*) + \alpha^*\delta^* = 1$, then $P_0(x) = \gamma^*(1-x) + \delta^*$, $0 \leqslant x \leqslant 1$ satisfies the following relations:

$$B_0^N P_0(0) = 1, \quad B_1^N P_0(1) = 0,$$

$P_1(x) = \alpha^* x + \beta^*$, $0 \leqslant x \leqslant 1$ satisfies the following relations:

$$B_0^N P_1(0) = 0, \quad B_1^N P_1(1) = 1.$$

Consider the barrier function $\Phi_i = m_0 P_0(x) + m_1 P_1(x) \pm u_i$, $0 \leqslant i \leqslant N$,
where $m_0 = \max\{K_2 + (m_1(\varepsilon\alpha^* \bar{p} + \alpha^* \bar{\beta}) + K_1)/(\gamma^* \underline{\beta})\}$, $m_1 \geqslant K_3$.
We have
$B_0^N \Phi_0 \geqslant 0$, $B_1^N \Phi_N \geqslant 0$, $L^N \Phi_i \leqslant 0$, $0 < i < N$.
By using Lemma 2 for the mesh function Φ_i, we yield the estimate $|u_i| \leqslant C$, for all $0 \leqslant i \leqslant N$.
We often use the following inequalities and equality in the below discussion:

$$C_1 \cdot t \leqslant \sinh t \leqslant C_2 \cdot t, \quad 0 < t < c, \tag{3.8}$$

$$C_1 \cdot \exp t \leqslant \sinh t \leqslant C_2 \cdot \exp t, \quad t \geqslant c > 0, \tag{3.9}$$

$$\left(D_+ - \frac{\mathrm{d}}{\mathrm{d}x}\right)\varphi(x_i) = \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_i+1} \varphi''(t)(x_{i+1} - t)\,\mathrm{d}t, \tag{3.10}$$

$$\left|\frac{1}{\varepsilon^k}\exp\left(-\frac{1}{\varepsilon}\right)\right| \leqslant C, \tag{3.11}$$

$$\left|\exp\left(-\frac{\mathrm{d}x_{N/2}}{\varepsilon}\right)\right| \leqslant CN^{-1}. \tag{3.12}$$

4. Uniform convergence of fitted mesh finite difference scheme

We define the following difference equations:

$$\begin{aligned} &L^N v_i = Lv(x_i), \quad 0 < i < N,\\ &B_0^N v_0 = B_0 v(0),\\ &B_1^N v_N = B_1 v(1) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &L^N z_i = Lz(x_i), \quad 0 < i < N,\\ &B_0^N z_0 = B_0 z(0),\\ &B_1^N z_N = B_1 z(1) \end{aligned} \tag{4.2}$$

then we have the decomposition $u_i = e \cdot v_i + z_i$.

Lemma 4. *Let v_i be the solution of difference equation* (4.1), *then* $|e \cdot (v(x_i) - v_i)| \leqslant CN^{-1} \ln N$ *for all* $0 \leqslant i \leqslant N$.

Proof.

$$L^N(v(x_i) - v_i) = L^N v(x_i) - Lv(x_i),$$

$$L^N(e(v(x_i) - v_i)) = eG_1 + eG_2, \tag{4.3}$$

where

$$G_1 = \varepsilon \cdot \delta(p(x_i) \cdot \delta v(x_i)) - \varepsilon(p(x)v'(x))'(x_i), \tag{4.4}$$

$$G_2 = D_+(q(x_i)v(x_i)) - (q(x)v(x))'(x_i). \tag{4.5}$$

Since

$$v(x_{i+1}) = v(x_{i+1/2}) + \frac{1}{2}h_{i+1}v'(x_{i+1/2}) + \frac{1}{2}\left(\frac{h_{i+1}}{2}\right)^2 v''(x_{i+1/2})$$
$$+ \frac{1}{2}\int_{x_i+1/2}^{x_i+1} v'''(t)(x_i + h_{i+1} - t)^2\,\mathrm{d}t$$

and

$$v(x_i) = v(x_{i+1/2}) - \frac{1}{2} h_{i+1} v'(x_{i+1/2}) + \frac{1}{2}\left(\frac{h_{i+1}}{2}\right)^2 v''(x_{i+1/2})$$
$$- \frac{1}{2}\int_{x_i}^{x_i+1/2} v'''(t)(t - x_i)^2 \, dt.$$

We obtain

$$\frac{v(x_{i+1}) - v(x_i)}{h_{i+1}} = v'(x_{i+1/2}) + F_{01}, \tag{4.6}$$

where

$$F_{01} = \frac{1}{2h_{i+1}}\left\{\int_{x_i+1/2}^{x_i+1} v'''(t)(x_i + h_{i+1} - t)^2 \, dt + \int_{x_i}^{x_i+1/2} v'''(t)(t - x_i)^2 \, dt\right\}. \tag{4.7}$$

Similarly,

$$\frac{v(x_i) - v(x_{i-1})}{h_i} = v'(x_{i-1/2}) + F_{02}, \tag{4.8}$$

where

$$F_{02} = \frac{1}{2h_i}\left\{\int_{x_i-1/2}^{x_i} v'''(t)(x_i - t)^2 \, dt + \int_{x_i-1}^{x_i-1/2} v'''(t)(t - x_i + h_i)^2 \, dt\right\}. \tag{4.9}$$

Thus, we get

$$G_1 = \frac{\varepsilon}{\bar{h}_i}[p(x_{i+1/2})v'(x_{i+1/2}) - p(x_{i-1/2})v'(x_{i-1/2}) + p(x_{i+1/2})F_{01} - p(x_{i-1/2})F_{02}]$$
$$- \varepsilon(p(x)v'(x))'(x_i).$$

By using Taylor expansion

$$p(x_{i+1/2}) = p(x_i) + \frac{h_{i+1}}{2} p'(x_i) + O(h_{i+1}^2),$$

$$v'(x_{i+1/2}) = v'(x_i) + \frac{h_{i+1}}{2} v''(x_i) + \int_{x_i}^{x_i+1/2} v'''(t)\left(x_i + \frac{h_{i+1}}{2} - t\right) dt.$$

We have

$$p(x_{i+1/2})v'(x_{i+1/2}) = p(x_i)v'(x_i) + \frac{h_{i+1}}{2} p(x_i)v''(x_i) + \frac{h_{i+1}}{2} p'(x_i)v'(x_i)$$
$$+ \frac{h_{i+1}^2}{4} p'(x_i)v''(x_i) + F_{03},$$

where

$$F_{03} = \left[p(x_i) + \frac{h_{i+1}}{2} p'(x_i)\right]\int_{x_i}^{x_i+1/2} v'''(t)\left(x_i + \frac{h_{i+1}}{2} - t\right) dt$$
$$+ O(h_{i+1}^2)v'(x_{i+1/2}). \tag{4.10}$$

Similarly,

$$p(x_{i-1/2})v'(x_{i-1/2}) = p(x_i)v'(x_i) - \frac{h_i}{2}\,p(x_i)v''(x_i) - \frac{h_i}{2}\,p'(x_i)v'(x_i) + \frac{h_i^2}{4}\,p'(x_i)v''(x_i) + F_{04},$$

where

$$F_{04} = \left[p(x_i) - \frac{h_i}{2}\,p'(x_i)\right]\int_{x_i-1/2}^{x_i} v'''(t)\left(t - x_i + \frac{h_i}{2}\right)\,\mathrm{d}t + \mathrm{O}(h_i^2)v'(x_{i-1/2}). \tag{4.11}$$

Therefore, we have

$$G_1 = \frac{\varepsilon}{\bar{h}_i}\,[p(x_{i+1/2})F_{01} - p(x_{i-1/2})F_{02}] + \frac{\varepsilon}{\bar{h}_i}\,[F_{03} - F_{04}]. \tag{4.12}$$

To estimate the local truncation error, the argument depends on the transition point σ. The corresponding piecewise uniform mesh is constructed by dividing both $[0,\sigma]$ and $[1-\sigma,1]$ into equal subintervals. The argument consists of following four cases.

Case 1: $0 < i < N/2$.

We have

$$\begin{aligned}
\left|e\,\frac{\varepsilon}{h_l}\,F_{01}\right| &\leqslant C\,\frac{\varepsilon^2}{h_l^2}\max_{x\in[x_i,x_{i+1}]} v'''(x)\left\{\int_{x_i+1/2}^{x_i+1}(x_i+h_l-t)^2\,\mathrm{d}t + \int_{x_i}^{x_i+1/2}(t-x_i)^2\,\mathrm{d}t\right\}\\
&\leqslant C\,\frac{\varepsilon^2}{h_l^2}\,\frac{1}{\varepsilon^3}\,h_l^3\\
&\leqslant C\,\frac{h_l}{\varepsilon}\\
&\leqslant CN^{-1}\ln N,
\end{aligned}$$

$$\begin{aligned}
\left|e\,\frac{\varepsilon}{h_l}\,F_{02}\right| &\leqslant C\,\frac{\varepsilon^2}{h_l^2}\max_{x\in[x_{i-1},x_i]} v'''(x)\left\{\int_{x_i-1/2}^{x_i}(x_i-t)^2\,\mathrm{d}t + \int_{x_i-1}^{x_i-1/2}(t-x_i+h_l)^2\,\mathrm{d}t\right\}\\
&\leqslant CN^{-1}\ln N,
\end{aligned}$$

$$\begin{aligned}
\left|e\,\frac{\varepsilon}{h_l}\,F_{03}\right| &\leqslant C\,\frac{\varepsilon^2}{h_l}\left\{\max_{x\in[x_i,x_{i+1/2}]} v'''(x)\int_{x_i}^{x_{i+1/2}}\left(x_i+\frac{h_l}{2}-t\right)\,\mathrm{d}t + h_l^2 v'(x_{i+1/2})\right\}\\
&\leqslant CN^{-1}\ln N,
\end{aligned}$$

$$\begin{aligned}
\left|e\,\frac{\varepsilon}{h_l}\,F_{04}\right| &\leqslant C\,\frac{\varepsilon^2}{h_l}\left\{\max_{x\in[x_{i-1/2},x_i]} v'''(x)\int_{x_i-1/2}^{x_i}\left(t-x_i+\frac{h_l}{2}\right)\,\mathrm{d}t + h_l^2 v'(x_{i-1/2})\right\}\\
&\leqslant CN^{-1}\ln N.
\end{aligned}$$

Therefore, we get

$$|eG_1| \leqslant CN^{-1} \ln N. \tag{4.13}$$

On the other hand,

$$G_2 = \frac{q(x_{i+1})v(x_{i+1}) - q(x_i)v(x_i)}{h_l} - (q(x)v(x))'(x_i).$$

Hence

$$G_2 = \frac{h_l}{2}(q(x)v(x))''(\xi) \quad \text{where } \xi \in (x_i, x_{i+1}). \tag{4.14}$$

It is easy to analyze the bound on G_2.

$$|eG_2| \leqslant C\varepsilon h_l \frac{1}{\varepsilon^2} \leqslant CN^{-1} \ln N. \tag{4.15}$$

Combining (4.13) and (4.15), (4.3) becomes

$$|L^N(e(v_i - v(x_i)))| \leqslant CN^{-1} \ln N. \tag{4.16}$$

Case 2: $N/2 < i < N$.

We have

$$\begin{aligned}
\left| e\frac{\varepsilon}{h_r} F_{01} \right| &\leqslant C \frac{\varepsilon^2}{h_r^2} h_r^2 \left\{ \int_{x_{i+1/2}}^{x_i+1} v'''(t)\,\mathrm{d}t + \int_{x_i}^{x_{i+1/2}} v'''(t)\,\mathrm{d}t \right\} \\
&\leqslant C\varepsilon^2 \int_{x_i}^{x_{i+1}} v'''(t)\,\mathrm{d}t \\
&\leqslant C \exp\left(-\frac{q(0)\cdot x_{i+1/2}}{p(0)\cdot\varepsilon} \right) sh\left(\frac{q(0)h_r}{2p(0)\varepsilon} \right) \\
&\leqslant C \exp\left(-\frac{q(0)\cdot x_{N/2}}{p(0)\cdot\varepsilon} \right) \exp\left(-\frac{q(0)h_r}{2p(0)\varepsilon} \right) sh\left(\frac{q(0)h_r}{2p(0)\varepsilon} \right).
\end{aligned}$$

When $h_r/\varepsilon \leqslant 1$,

$$\left| \exp\left(-\frac{q(0)h_r}{2p(0)\varepsilon} \right) sh\left(\frac{q(0)h_r}{2p(0)\varepsilon} \right) \right| \leqslant C$$

hold by using inequalities (3.8) and (3.11).

When $h_r/\varepsilon > 1$,

$$\left| \exp\left(-\frac{q(0)h_r}{2p(0)\varepsilon} \right) sh\left(\frac{q(0)h_r}{2p(0)\varepsilon} \right) \right| \leqslant C$$

hold by using inequality (3.9).

Therefore, we obtain

$$\left| e\frac{\varepsilon}{h_r} F_{01} \right| \leqslant CN^{-1}. \tag{4.17}$$

Similarly,

$$\left| e\,\frac{\varepsilon}{h_r}\,F_{02} \right| \leqslant CN^{-1}, \tag{4.18}$$

$$\begin{aligned}\left| e\,\frac{\varepsilon}{h_r}\,F_{03} \right| &\leqslant C\,\frac{\varepsilon^2}{h_r}\left\{ h_r \int_{x_i}^{x_{i+1/2}} v'''(t)\,\mathrm{d}t + h_r^2 v'(x_{i+1/2}) \right\} \\ &\leqslant CN^{-1},\end{aligned}$$

$$\begin{aligned}\left| e\,\frac{\varepsilon}{h_r}\,F_{04} \right| &\leqslant C\,\frac{\varepsilon^2}{h_r}\left\{ h_r \int_{x_{i-1/2}}^{x_i} v'''(t)\,\mathrm{d}t + h_r^2 v'(x_{i-1/2}) \right\} \\ &\leqslant CN^{-1}.\end{aligned}$$

Thus, we obtain

$$|eG_1| \leqslant CN^{-1}. \tag{4.19}$$

On the other hand, we obtain by using equality (3.10)

$$\begin{aligned} G_2 &= \frac{q(x_{i+1})v(x_{i+1}) - q(x_i)v(x_i)}{h_r} - (q(x)v(x))'(x_i) \\ &= \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} (q(t)v(t))''(x_{i+1} - t)\,\mathrm{d}t. \end{aligned}$$

Hence, we have

$$|eG_2| \leqslant CN^{-1}. \tag{4.20}$$

Therefore, we obtain

$$|L^N(e(v_i - v(x_i)))| \leqslant CN^{-1}. \tag{4.21}$$

hold when $N/2 < i < N$.

Case 3: $i = N/2$.

we obtain by using Taylor expansion

$$\frac{v(x_{N/2+1}) - v(x_{N/2})}{h_r} = v'(x_{N/2+1/2}) + F_{05}$$

and

$$\frac{v(x_{N/2}) - v(x_{N/2-1})}{h_l} = v'(x_{N/2-1/2}) + F_{06},$$

where

$$F_{05} = \frac{1}{2h_r}\left\{ \int_{x_{N/2+1/2}}^{x_{N/2+1}} v'''(t)(x_{N/2} + h_r - t)^2\,\mathrm{d}t + \int_{x_{N/2}}^{x_{N/2+1/2}} v'''(t)(t - x_{N/2})^2\,\mathrm{d}t \right\}, \tag{4.22}$$

$$F_{06}=\frac{1}{2h_l}\left\{\int_{x_{N/2-1/2}}^{x_{N/2}} v'''(t)(x_{N/2}-t)^2\,\mathrm{d}t+\int_{x_{N/2-1}}^{x_{N/2-1/2}} v'''(t)(t-x_{N/2}+h_l)^2\,\mathrm{d}t\right\}. \tag{4.23}$$

By using Taylor expansion, we obtain

$$G_1=\frac{\varepsilon}{\bar{h}}\,[p(x_{N/2+1/2})F_{05}-\,p(x_{N/2-1/2})F_{06}]+\frac{\varepsilon}{\bar{h}}\,[F_{07}-F_{08}+F_{09}], \tag{4.24}$$

where

$$F_{07}=[p(x_{N/2})+\frac{h_r}{2}\,p'(x_{N/2})]\int_{x_{N/2}}^{x_{N/2+1/2}} v'''(t)\left(x_{N/2}+\frac{h_r}{2}-t\right)\mathrm{d}t$$

$$+\,\mathrm{O}(h_r^2)v'(x_{N/2+1/2}), \tag{4.25}$$

$$F_{08}=\left[p(x_{N/2})-\frac{h_l}{2}\,p'(x_{N/2})\right]\int_{x_{N/2-1/2}}^{x_{N/2}} v'''(t)\left(t-x_{N/2}+\frac{h_l}{2}\right)\mathrm{d}t$$

$$+\,\mathrm{O}(h_l^2)v'(x_{N/2-1/2}), \tag{4.26}$$

$$F_{09}=\bar{h}\,\frac{h_r-h_l}{2}\,p'(x_{N/2})v''(x_{N/2}). \tag{4.27}$$

Similar to the argument in Cases 1 and 2, we find the estimates

$$\left|e\,\frac{\varepsilon}{\bar{h}}\,F_{05}\right|\leqslant CN^{-1},$$

$$\left|e\,\frac{\varepsilon}{\bar{h}}\,F_{16}\right|\leqslant CN^{-1}\ln N,$$

$$\left|e\,\frac{\varepsilon}{\bar{h}}\,F_{07}\right|\leqslant CN^{-1},$$

$$\left|e\,\frac{\varepsilon}{\bar{h}}\,F_{08}\right|\leqslant CN^{-1}\ln N,$$

$$\left|e\,\frac{\varepsilon}{\bar{h}}\,F_{09}\right|\leqslant CN^{-1}.$$

Therefore, we have the estimate

$$|eG_1|\leqslant CN^{-1}\ln N. \tag{4.28}$$

Similar to case 2, by using inequality (3.10) we have the estimate

$$|eG_2|\leqslant CN^{-1}. \tag{4.29}$$

Therefore, the estimate

$$|L^N(e(v_i - v(x_i)))| \leqslant CN^{-1}\ln N \tag{4.30}$$

hold, when $i = N/2$.

Case 4: $i = 0$ *or* $i = N$.

The following identities hold:

$$\begin{aligned} B_0^N(v(0) - v_0) &= B_0^N v(0) - B_0 v(0) \\ &= -\beta^* h_l v''(\xi) \quad \text{where } 0 < \xi < h_l \end{aligned}$$

and

$$\begin{aligned} B_1^N(v(1) - v_N) &= B_1^N v(1) - B_1 v(1) \\ &= \delta^* h_r v''(\eta) \quad \text{where } 1 - h_r < \eta < 1. \end{aligned}$$

Hence, we have the estimates

$$\begin{aligned} |B_0^N(e(v(0) - v_0))| &\leqslant C\varepsilon h_l \frac{1}{\varepsilon^2} \\ &\leqslant CN^{-1}\ln N, \end{aligned}$$

$$\begin{aligned} |B_1^N(e(v(1) - v_N))| &\leqslant C\varepsilon h_r \\ &\leqslant CN^{-1}. \end{aligned}$$

By using Lemma 3, we yield the estimate

$$|e(v(x_i) - v_i)| \leqslant CN^{-1}\ln N.$$

In the case $(\varepsilon \ln N)/d \geqslant 1/2$, the mesh is uniform, so $h_l = h_r = h$. We notice that $1/\varepsilon \leqslant C\ln N$. Thus, we have the estimates

$$|eG_1| \leqslant CN^{-1}\ln N,$$

$$|eG_2| \leqslant CN^{-1}\ln N.$$

Hence, we find the estimate

$$|L^N(e(v_i - v(x_i)))| \leqslant CN^{-1}\ln N.$$

For the boundary conditions, we also have the estimate

$$|B_0^N(e(v(0) - v_0))| \leqslant CN^{-1}\ln N,$$

$$|B_1^N(e(v(1) - v_N))| \leqslant CN^{-1}.$$

By using Lemma 3 again, we obtain

$$|e(v(x_i) - v_i)| \leqslant CN^{-1}\ln N.$$

Lemma 5. *Let z_i be the solution of difference equation (4.2), then $|z(x_i) - z_i| \leqslant CN^{-1} \ln N$ for all $0 \leqslant i \leqslant N$.*

Proof.

$$L^N(z(x_i) - z_i) = L^N z(x_i) - Lz(x_i) = G_3 + G_4, \tag{4.31}$$

where

$$G_3 = \varepsilon \cdot \delta(p(x_i) \cdot \delta z(x_i)) - \varepsilon(p(x)z'(x))'(x_i), \tag{4.32}$$

$$G_4 = D_+(q(x_i)z(x_i)) - (q(x)z(x))'(x_i). \tag{4.33}$$

Furthermore, we have

$$G_3 = \frac{\varepsilon}{\bar{h}_i}[p(x_{i+1/2})F_{11} - p(x_{i-1/2})F_{12}] + \frac{\varepsilon}{\bar{h}_i}[F_{13} - F_{14}], \tag{4.34}$$

where

$$F_{11} = \frac{1}{2h_{i+1}} \left\{ \int_{x_i+1/2}^{x_i+1} z'''(t)(x_i + h_{i+1} - t)^2 \, dt + \int_{x_i}^{x_i+1/2} z'''(t)(t - x_i)^2 \, dt \right\}, \tag{4.35}$$

$$F_{12} = \frac{1}{2h_i} \left\{ \int_{x_i-1/2}^{x_i} z'''(t)(x_i - t)^2 \, dt + \int_{x_i-1}^{x_i-1/2} z'''(t)(t - x_i + h_i)^2 \, dt \right\}, \tag{4.36}$$

$$F_{13} = \left[p(x_i) + \frac{h_{i+1}}{2} p'(x_i)\right] \int_{x_i}^{x_i+1/2} z'''(t) \left(x_i + \frac{h_{i+1}}{2} - t\right) dt + O(h_{i+1}^2)z'(x_{i+1/2}), \tag{4.37}$$

$$F_{14} = \left[p(x_i) - \frac{h_i}{2} p'(x_i)\right] \int_{x_i-1/2}^{x_i} z'''(t) \left(t - x_i + \frac{h_i}{2}\right) dt + O(h_i^2)z'(x_{i-1/2}). \tag{4.38}$$

Firstly, we discuss the piecewise uniform mesh with $\varepsilon \ln N/d < 1/2$.

When $1 \leqslant i \leqslant N/2 - 1$, a computation gives

$$\begin{aligned}
\left|\frac{\varepsilon}{h_l} F_{11}\right| &\leqslant C \frac{\varepsilon}{h_l^2} \max_{x \in [x_i, x_{i+1}]} z'''(x) \left\{ \int_{x_i+1/2}^{x_i+1} (x_i + h_l - t)^2 \, dt + \int_{x_i}^{x_i+1/2} (t - x_i)^2 \, dt \right\} \\
&\leqslant C \frac{\varepsilon}{h_l^2} \left(1 + \frac{1}{\varepsilon^2}\right) h_l^3 \\
&\leqslant CN^{-1} + C \frac{h_l}{\varepsilon} \\
&\leqslant CN^{-1} \ln N
\end{aligned}$$

and

$$\left|\frac{\varepsilon}{h_l} F_{1i}\right| \leqslant CN^{-1} \ln N, \quad 2 \leqslant i \leqslant 4.$$

Hence, we have

$$|G_3| \leqslant CN^{-1}\ln N, \quad \text{hold for } 1 \leqslant i \leqslant \frac{N}{2}-1.$$

Furthermore, we have

$$|G_4| \leqslant CN^{-1}\ln N, \quad \text{hold for } 1 \leqslant i \leqslant \frac{N}{2}-1.$$

Hence, we find the estimate

$$|L^N(z_i - z(x_i))| \leqslant CN^{-1}\ln N, \quad \text{hold for } 1 \leqslant i \leqslant \frac{N}{2}-1.$$

Similarly, we have the estimate

$$|L^N(z_i - z(x_i))| \leqslant CN^{-1}\ln N, \quad \text{hold for } \frac{N}{2} \leqslant i \leqslant N-1.$$

The following identities hold:

$$\begin{aligned} B_0^N(z(0)-z_0) &= B_0^N z(0) - B_0 z(0) \\ &= -\beta^* h_l z''(\xi), \quad \text{where } 0 < \xi < h_l \end{aligned}$$

and

$$\begin{aligned} B_1^N(z(1)-z_N) &= B_1^N z(1) - B_1 z(1) \\ &= \delta^* h_r z''(\eta), \quad \text{where } 1-h_r < \eta < 1. \end{aligned}$$

Hence, we have the estimates

$$\begin{aligned} |B_0^N(z(0)-z_0)| &\leqslant Ch_l\left(1+\frac{1}{\varepsilon}\right) \\ &\leqslant CN^{-1} + CN^{-1}\ln N \\ &\leqslant CN^{-1}\ln N, \end{aligned}$$

$$|B_1^N(z(1)-z_N)| \leqslant CN^{-1}.$$

By using Lemmas 3, we yield the estimate

$$|z(x_i)-z_i| \leqslant CN^{-1}\ln N.$$

In the case $\varepsilon \ln N/d \geqslant \frac{1}{2}$, the mesh is uniform, we also have the result

$$|z(x_i)-z_i| \leqslant CN^{-1}\ln N.$$

By combining Lemmas 4 and 5, we obtain the following theorem.

Theorem 1. *Let $u(x)$ be the solution of SPMBP and u_i be the solution of FMFDS, then $|u(x_i)-u_i| \leqslant CN^{-1}\ln N$ for all $0 \leqslant i \leqslant N$.*

5. A class of conservative difference schemes of SPMBP

Let [0,1] be divided into N uniformly spaced mesh intervals, with mesh spacing $h=N^{-1}$ and with mesh points $x_i=i\cdot h$, $0\leqslant i\leqslant N$. We define a class of conservative difference schemes (CCDS) of SPMBP by

$$L^h u_i \equiv \varepsilon\cdot\delta(\sigma_i p_{i-s}\cdot\delta u_i)+D_0(q_{i-t+1/2}u_i)+r(x_i)u_i=f(x_i),\quad 1\leqslant i\leqslant N-1, \tag{5.1}$$

$$B_0^h u_0 \equiv \alpha^* u_0-\beta^*\frac{u_1-u_0}{h}=A, \tag{5.2}$$

$$B_1^h u_N \equiv \gamma^* u_N+\delta^*\frac{u_N-u_{N-1}}{h}=B, \tag{5.3}$$

where

$$\delta u_i=\frac{u_{i+1/2}-u_{i-1/2}}{h},\quad D_0 u_i=\frac{u_{i+1}-u_{i-1}}{2h},$$

$$p_{i-s}=p(x_{i-s}), q_{i-t+1/2}=q(x_{i-t+1/2}),$$

$$\sigma_i=\sigma(x_i,s,t,\rho)=R(x_i,s,t)\rho\cdot\coth(R(x_i,s,t)\rho),$$

$$R(x,s,t)=\frac{q(x-t\cdot h)}{2p(x-s\cdot h)}>d>0,\quad \rho=\frac{h}{\varepsilon},$$

$$x\in[h,1-h],\quad s\in[0,0.5],\quad t\in[0,0.5].$$

Lemma 6. *If the mesh function u_i satisfies $L^h u_i\leqslant 0$, $B_0^h u_0\geqslant 0$, $B_1^h u_N\geqslant 0$, then $u_i\geqslant 0$ for all $0\leqslant i\leqslant N$.*

Proof. Analogous to the proof on Lemma 2.

Corresponding to [2,16], we have the following discrete maximum principle.

Lemma 7. *If the mesh function u_i satisfies*

$$|L^h u_i|\leqslant C\cdot\left\{1+\frac{C_0}{\max(h,\varepsilon)}\exp\left(-\frac{d\cdot x_{i-1}}{\varepsilon}\right)\right\},\quad 1\leqslant i\leqslant N-1, \tag{5.4}$$

$$|B_0^h u_0|\leqslant C_1\cdot\beta^*\left\{1+\frac{1-\exp(-\bar{d}\cdot\rho)}{h}\right\}, \tag{5.5}$$

$$|B_1^h u_N|\leqslant C_2 \tag{5.6}$$

then $|u_i|\leqslant C$ for all $0\leqslant i\leqslant N$.

We often use following inequalities and equality in the below discussion:

$$|t\cdot\coth t-1|\leqslant c\cdot t,\quad t\in(0,+\infty), \tag{5.7}$$

$$|t \cdot \coth t - 1| \leqslant \frac{c \cdot t^2}{1+t}, \quad t \in (0, +\infty), \tag{5.8}$$

$$|1 - t \cdot \sinh^{-2} t| \leqslant c \cdot t^2, \quad t \in (0, +\infty), \tag{5.9}$$

$$\left|\frac{\partial \sigma}{\partial x}\right| \leqslant c \cdot \rho, \quad t \in (0, +\infty), \tag{5.10}$$

$$\sinh t = t + s, \quad \text{where } |s| \leqslant \frac{c \cdot |t|^3}{1+|t|^2} \exp t, \quad 0 < t < c, \tag{5.11}$$

$$\varepsilon \cdot \left|\frac{\partial \sigma}{\partial x}\right| \leqslant c \cdot \frac{h^2}{h+\varepsilon}, \quad \text{where } h \leqslant \varepsilon. \tag{5.12}$$

6. Uniform convergence of CCDS

We define the following difference equations:

$$\begin{aligned} &L^h v_i = Lv(x_i), \quad 0 < i < N, \\ &B_0^h v_0 = B_0 v(0), \\ &B_1^h v_N = B_1 v(1) \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &L^h z_i = Lz(x_i), \quad 0 < i < N, \\ &B_0^h z_0 = B_0 z(0), \\ &B_1^h z_N = B_1 z(1) \end{aligned} \tag{6.2}$$

then we have the decomposition $u_i = e \cdot v_i + z_i$.

Lemma 8. *Let v_i be the solution of difference equation* (6.1), *then* $|e \cdot (v(x_i) - v_i)| \leqslant Ch$ *for all* $0 \leqslant i \leqslant N$.

Proof. Let

$$\begin{aligned} &T_0 = \frac{q(0)}{p(0)}, \\ &T(x_i) = \frac{q(x_i - t \cdot h - 0.5 \cdot h)}{p(x_i - s \cdot h - 0.5 \cdot h)}, \quad 1 \leqslant i \leqslant N-1. \end{aligned}$$

By using Taylor expansion, we have

$$p(x_{i-s+1/2}) = p(x_{i-s-1/2}) + h \cdot p'(x_{i-s-1/2}) + \frac{1}{2} h^2 \cdot p''(x_{i-s-1/2}) + \frac{h^3}{6} p'''(\xi_1),$$

$$q(x_{i-t+3/2}) = q(x_{i-t-1/2}) + 2h \cdot q'(x_{i-t-1/2}) + 2h^2 \cdot q''(x_{i-t-1/2}) + \frac{4h^3}{3} q'''(\zeta_1),$$

$$p(x_i) = p(x_{i-s-1/2}) + \left(s + \frac{1}{2}\right) \cdot h \cdot p'(x_{i-s-1/2}) + \frac{1}{2}\left(s + \frac{1}{2}\right)^2 \cdot h^2 \cdot p''(\xi_2),$$

$$q(x_i) = q(x_{i-t-1/2}) + \left(t + \frac{1}{2}\right) h \cdot q'(x_{i-t-1/2}) + \frac{1}{2}\left(t + \frac{1}{2}\right)^2 h^2 \cdot q''(x_{i-t-1/2})$$
$$+ \frac{(t+1/2)^3 \cdot h^3}{6} q'''(\zeta_2),$$

$$p'(x_i) = p'(x_{i-s-1/2}) + \left(s + \frac{1}{2}\right) \cdot h \cdot p''(x_{i-s-1/2}) + \frac{1}{2}\left(s + \frac{1}{2}\right)^2 \cdot h^2 \cdot p'''(\xi_3),$$

$$q'(x_i) = q'(x_{i-t-1/2}) + \left(t + \frac{1}{2}\right) \cdot h \cdot q''(x_{i-t-1/2}) + \frac{1}{2}\left(t + \frac{1}{2}\right)^2 \cdot h^2 \cdot q'''(\zeta_3)$$

where

$$ih - sh - 0.5h \leqslant \xi_1 \leqslant ih - sh + 0.5h, \quad ih - sh - 0.5h \leqslant \xi_2, \quad \xi_3 \leqslant ih,$$

$$ih - th - 0.5h \leqslant \zeta_1 \leqslant ih - th + 1.5h, \quad ih - th - 0.5h \leqslant \zeta_2, \quad \zeta_3 \leqslant ih.$$

Hence, we obtain

$$\begin{aligned}
L^h v(x_i) &= \frac{\varepsilon}{h^2}\{\sigma_{i+1/2} p_{i-s+1/2} \cdot [v(x_{i+1}) - v(x_i)] \\
&\quad - \sigma_{i-1/2} p_{i-s-1/2} \cdot [v(x_i) - v(x_{i-1})]\} \\
&\quad + \frac{1}{2h}\{q_{i-t+3/2} \cdot v(x_{i+1}) - q_{i-t-1/2} \cdot v(x_{i-1})\} + r(x_i)v(x_i) \\
&= \frac{\varepsilon}{h^2} p_{i-s+1/2} \cdot [\sigma_{i+1/2} - \sigma_{i-1/2}][v(x_{i+1}) - v(x_i)] \\
&\quad - 2q(x_{i-t-1/2}) \cdot \frac{\sinh \frac{T_0\rho}{2} \sinh \frac{[T(x_i) - T(x_0)]\rho}{2}}{h \cdot \sinh \frac{T(x_i)}{2}\rho} v(x_i) \\
&\quad + \left[\frac{\sigma_{i-1/2}}{\rho} p'(x_{i-s-1/2}) + \frac{1}{2}\varepsilon\sigma_{i-1/2} p''(x_{i-s-1/2})\right. \\
&\quad \left. + \frac{\varepsilon\sigma_{i-1/2} h}{6} p'''(\xi_1)\right][v(x_{i+1}) - v(x_i)] \\
&\quad + [q'(x_{i-t-1/2}) + h \cdot q''(x_{i-t-1/2}) + \frac{2h^2}{3} q'''(\zeta_1)] \cdot v(x_{i+1}) + r(x_i)v(x_i)
\end{aligned}$$

and

$$Lv(x_i) = \left\{\frac{T_0^2}{\varepsilon} \cdot p(x_i) - \frac{T_0}{\varepsilon} \cdot q(x_i) - T_0 \cdot p'(x_i) + q'(x_i) + r(x_i)\right\} \cdot v(x_i)$$

$$= \left\{ \frac{T_0}{\varepsilon} \cdot p(x_{i-s-1/2})[T_0 - T(x_i)] + T_0 p'(x_{i-s-1/2}) \left[T_0 \rho \left(s + \frac{1}{2} \right) - 1 \right] \right.$$

$$- T_0 \left(s + \frac{1}{2} \right) h \cdot p''(x_{i-s-1/2}) + q'(x_{i-t-1/2}) \left[1 - T_0 \cdot \left(t + \frac{1}{2} \right) \rho \right]$$

$$+ q''(x_{i-t-1/2}) \left(t + \frac{1}{2} \right) h \cdot \left[1 - \frac{1}{2} T_0 \cdot \left(t + \frac{1}{2} \right) \rho \right]$$

$$+ \left[\frac{T_0^2}{2\varepsilon} \left(s + \frac{1}{2} \right)^2 h^2 \cdot p''(\xi_2) - \frac{T_0}{6\varepsilon} \left(t + \frac{1}{2} \right)^3 h^3 \cdot q'''(\zeta_2) \right.$$

$$\left. \left. - \frac{1}{2} T_0 \left(s + \frac{1}{2} \right)^2 h^2 \cdot p'''(\xi_3) + \frac{1}{2} \left(t + \frac{1}{2} \right)^2 h^2 \cdot q'''(\zeta_3) \right] + r(x_i) \right\} \cdot v(x_i).$$

Therefore, we have

$$L^h(v_i - v(x_i)) = Lv(x_i) - L^h v(x_i) = \sum_{i=1}^{7} F_i, \tag{6.3}$$

where

$$F_1 = -\frac{\varepsilon}{h^2} p(x_{i-s+1/2}) \cdot [\sigma_{i+1/2} - \sigma_{i-1/2}][v(x_{i+1}) - v(x_i)],$$

$$F_2 = \left\{ \frac{T_0}{\varepsilon} p(x_{i-s-1/2}) \cdot [T_0 - T(x_i)] \right.$$
$$\left. + 2q(x_{i-t-1/2}) \cdot \frac{\sinh T_0\rho/2 \cdot \sinh(T(x_i) - T_0)\rho/2}{h \cdot \sinh T(x_i)\rho/2} \right\} v(x_i),$$

$$F_3 = p'(x_{i-s-1/2}) \left\{ \left[T_0^2 \rho \left(s + \frac{1}{2} \right) - T_0 \right] v(x_i) - \frac{\sigma_{i-1/2}}{\rho} \cdot [v(x_{i+1}) - v(x_i)] \right\},$$

$$F_4 = p''(x_{i-s-1/2}) \left\{ -T_0 \left(s + \frac{1}{2} \right) \cdot h \cdot v(x_i) - \frac{1}{2} \varepsilon \sigma_{i-1/2}[v(x_{i+1}) - v(x_i)] \right\},$$

$$F_5 = q'(x_{i-t-1/2}) \left\{ \left[1 - T_0 \left(t + \frac{1}{2} \right) \rho \right] v(x_i) - v(x_{i+1}) \right\},$$

$$F_6 = q''(x_{i-t-1/2}) h \left\{ t + \frac{1}{2} - \frac{1}{2} T_0 \left(t + \frac{1}{2} \right)^2 \rho - \exp(-T_0\rho) \right\} v(x_i),$$

$$F_7 = \left[\frac{T_0^2}{2\varepsilon}\left(s+\frac{1}{2}\right)^2 h^2 \cdot p''(\xi_2) - \frac{T_0}{6\varepsilon}\left(t+\frac{1}{2}\right)^3 h^3 \cdot q'''(\zeta_2)\right.$$
$$\left. - \frac{1}{2} T_0 \left(s+\frac{1}{2}\right)^2 h^2 \cdot p'''(\xi_3) + \frac{1}{2}\left(t+\frac{1}{2}\right)^2 h^2 q'''(\zeta_3)\right] \cdot v(x_i)$$
$$- \frac{\varepsilon \cdot \sigma_{i-1/2} \cdot h}{6} \cdot p'''(\xi_1)[v(x_{i+1}) - v(x_i)] - \frac{2h^2}{3} q'''(\zeta_1) v(x_{i+1}).$$

Above equality consists of 7 terms. We can estimate the bound of each term as follows.

(i) From inequality (5.10) we get

$$|F_1| \leqslant Ch \frac{1}{\max(h,\varepsilon)} \cdot \exp\left(-\frac{d \cdot x_{i-1}}{\varepsilon}\right).$$

(ii) F_2 may be written as

$$F_2 = v(x_i) \cdot p(x_{i-s-1/2})$$
$$\times \frac{T_0\rho[T_0 - T(x_i)] \cdot \sinh \frac{T(x_i)\rho}{2} + 2T(x_i) \sinh \frac{T_0\rho}{2} \sinh \frac{[T(x_i)-T_0]\rho}{2}}{h \cdot \sinh \frac{T(x_i)\rho}{2}}.$$

By mean of [5, Lemma 4.3], it is easy to find a constant C_1, so that we have the estimate

$$|F_2| \leqslant Ch \frac{1}{\max(h,\varepsilon)} \cdot \exp\left(-\frac{d \cdot x_{i-1}}{\varepsilon}\right), 1 \leqslant i \leqslant N-1, \quad \text{when } h \leqslant C_1.$$

(iii) When $h \leqslant \varepsilon$, we have

$$F_3 = p'(x_{i-s-1/2}) \left\{ \left[T_0^2 \rho \left(s+\frac{1}{2}\right) - T_0\right] - \frac{1}{\rho} \cdot [\exp(-T_0\rho) - 1]\right.$$
$$\left. + \frac{\mathrm{O}(\rho)}{\rho} \cdot [\exp(-T_0\rho) - 1]\right\} \cdot v(x_i),$$
$$= p'(x_{i-s-1/2}) \{T_0^2 \rho \cdot s + \mathrm{O}(\rho) + \mathrm{O}(\rho^2)\} v(x_i).$$

Hence, we find the estimate

$$|F_3| \leqslant C\rho v(x_i) \leqslant C \cdot \frac{h}{\varepsilon} \exp\left(-\frac{d \cdot x_{i-1}}{\varepsilon}\right).$$

When $h > \varepsilon$, we have the estimate

$$|F_3| \leqslant C \cdot \frac{h}{h} \cdot \exp\left(-\frac{d \cdot x_{i-1}}{\varepsilon}\right).$$

(iv) A computation gives

$$|F_4| \leqslant Ch \left\{1 + \frac{1}{\max(h,\varepsilon)} \exp\left(-\frac{d \cdot x_{i-1}}{\varepsilon}\right)\right\}.$$

(v) When $h \leqslant \varepsilon$, we obtain the estimate

$$|F_5| \leqslant C\rho v(x_i) \leqslant C \cdot \frac{h}{\varepsilon} \exp\left(-\frac{d \cdot x_{i-1}}{\varepsilon}\right).$$

When $h > \varepsilon$, we obtain the estimate

$$|F_5| \leqslant C \cdot v(x_{i-1}) \leqslant C \frac{h}{h} \exp\left(-\frac{d \cdot x_{i-1}}{\varepsilon}\right).$$

(vi) When $h \leqslant \varepsilon$, we have

$$|F_6| \leqslant Ch \cdot v(x_i).$$

When $h > \varepsilon$, we have

$$|F_6| \leqslant Ch \cdot v(x_{i-1}).$$

(vii) A computation gives

$$|F_7| \leqslant Ch \left\{1 + \frac{1}{\max(h, \varepsilon)} \exp\left(-\frac{d \cdot x_{i-1}}{\varepsilon}\right)\right\}.$$

Therefore, we have the following estimate:

$$|L^h(v_i - v(x_i))| \leqslant Ch \cdot \left\{1 + \frac{1}{\max(h, \varepsilon)} \exp\left(-\frac{d \cdot x_{i-1}}{\varepsilon}\right)\right\}, \quad \text{when } h \leqslant C_1.$$

It is easy to show

$$|B_0^h[e(v(0) - v_0)]| \leqslant C \cdot \beta^* \{1 - \exp(-\bar{d}\rho)\} \tag{6.4}$$

and

$$|B_1^h[e(v(1) - v_N)]| \leqslant Ch. \tag{6.5}$$

By using Lemma 7, we have the estimate

$$|e \cdot (v(x_i) - v_i)| \leqslant C \cdot h, \quad \text{when } h \leqslant C_1. \tag{6.6}$$

When $h \geqslant C_1$, it can be seen that $|e \cdot v_i| \leqslant C$ and $|e \cdot v(x_i)| \leqslant C$.

Thus, we obtain the estimate

$$|e \cdot (v(x_i) - v_i)| \leqslant C \cdot h, \quad \text{when } h \geqslant C_1. \tag{6.7}$$

By combining (6.6) and (6.7), the Lemma 8 is proved. □

Lemma 9. *Let z_i be the solution of difference equation (6.2), then $|z(x_i) - z_i| \leqslant C \cdot h$ for all $0 \leqslant i \leqslant N$.*

Proof. Let

$$G_1 = \varepsilon \cdot \delta((\sigma_i - 1)p_{i-s} \cdot \delta z(x_i)),$$

$$G_2 = \varepsilon \cdot \{\delta(p_{i-s} \cdot \delta z(x_i)) - (p(x)z'(x))'(x_i)\},$$

$$G_3 = D_0(q_{i-t+1/2} z(x_i)) - (q(x)z(x))'(x_i).$$

Hence, we obtain

$$L^h z(x_i) - Lz(x_i) = G_1 + G_2 + G_3.$$

We can estimate the bound of each G_i $(i = 1,2,3)$ as follows:

(i) By the equality in [6]

$$\delta(g(x)\delta k(x)) = g(x+0.5h)\cdot\delta^2 k(x) + \frac{k(x)-k(x-h)}{h^2}\int_{x-0.5h}^{x+0.5h} g'(t)\,\mathrm{d}t.$$

We obtain

$$\begin{aligned}
&\delta((\sigma_i - 1)\cdot p_{i-s}\cdot \delta z(x_i))\\
&\quad\leqslant C\cdot|1-\sigma_{i+1/2}|\cdot|\delta^2 z(x_i)|\\
&\qquad + C\cdot \max_{x\in[x_{i-1},x_i]}|z'(x)|\left\{|\sigma(\xi_1,s,t,\rho)-1| + \left|\frac{\partial\sigma(\xi_2,s,t,\rho)}{\partial x}\right|\right\},
\end{aligned}$$

where

$$\xi_1,\xi_2\in(x-0.5h, x+0.5h).$$

Since

$$\begin{aligned}
|\delta^2 z(x_i)| &\leqslant Ch^{-1}\int_{x_i-h}^{x_i+h}|z''(x)|\,\mathrm{d}x\\
&\leqslant C\cdot\left\{1+\frac{1}{\max(h,\varepsilon)}\exp\left(-\frac{d\cdot x_{i-1}}{\varepsilon}\right)\right\}.
\end{aligned}$$

By using inequalities (5.7) and (5.10), we obtain the estimate

$$|G_1| \leqslant Ch\left\{1+\frac{1}{\max(h,\varepsilon)}\exp\left(-\frac{d\cdot x_{i-1}}{\varepsilon}\right)\right\}.$$

(ii) Analogous to discussion in Lemma 4, we obtain

$$\begin{aligned}
G_2 = \varepsilon\cdot\Bigg\{&\frac{1}{2h}\Bigg[p(x_{i-s+1/2})\int_{x_i}^{x_i+1/2} z^{(4)}(s)\left(x_i+\frac{h}{2}-s\right)^2\mathrm{d}s\\
&+ p(x_{i-s-1/2})\int_{x_i-1/2}^{x_i} z^{(4)}(s)\left(s-x_i+\frac{h}{2}\right)^2\mathrm{d}s\Bigg]\\
&+\frac{1}{2h^2}\cdot p(x_{i-s+1/2})\left[\int_{x_i+1/2}^{x_i+1} z'''(s)(x_i+h-s)^2\mathrm{d}s + \int_{x_i}^{x_i+1/2} z'''(s)(s-x_i)^2\,\mathrm{d}s\right]\\
&-\frac{1}{2h^2}\cdot p(x_{i-s-1/2})\left[\int_{x_i-1/2}^{x_i-1} z'''(s)(x_i-s)^2\,\mathrm{d}s + \int_{x_i-1}^{x_i-1/2} z'''(s)(s-x_i+h)^2\,\mathrm{d}s\right]\\
&+\mathrm{O}(h)z'(x_i)+\mathrm{O}(h)z''(x_i)+\mathrm{O}(h^2)z'''(x_i)\Bigg\}.
\end{aligned}$$

Table 1
FMFDS, computed maximum pointwise error

ε	Number of intervals N			
	16	32	64	128
2^{0}	0.04955219	0.02502954	0.01257936	0.00630599
2^{-1}	0.06896792	0.03458401	0.01731596	0.00866383
2^{-2}	0.07272705	0.03598751	0.01789754	0.00892446
2^{-3}	0.06161129	0.03039256	0.01510644	0.00753257
2^{-4}	0.05745273	0.02881511	0.01443905	0.00722830
2^{-5}	0.03979587	0.02518979	0.01460125	0.00731122
2^{-6}	0.01794153	0.01182926	0.00732080	0.00435183
2^{-7}	0.00609976	0.00456523	0.00302856	0.00188210
2^{-8}	0.00558843	0.00268170	0.00128988	0.00062183
2^{-9}	0.00604826	0.00294743	0.00144214	0.00070750
2^{-10}	0.00628776	0.00308709	0.00152282	0.00075323
2^{-11}	0.00640999	0.00315869	0.00156437	0.00077689
2^{-12}	0.00647173	0.00319495	0.00158546	0.00078893
2^{-13}	0.00650276	0.00321319	0.00159609	0.00079501
2^{-14}	0.00651832	0.00322234	0.00160142	0.00079806
2^{-15}	0.00652611	0.00322693	0.00160409	0.00079959
2^{-16}	0.00653000	0.00322922	0.00160543	0.00080035
—	—	—	—	—
—	—	—	—	—
2^{-36}	0.00653390	0.00323151	0.00160676	0.00080112

Thus, we have the estimate

$$|G_2| \leqslant Ch\left\{1+\frac{1}{\max(h,\varepsilon)}\exp\left(-\frac{d\cdot x_{i-1}}{\varepsilon}\right)\right\}.$$

(iii) Analogous to discussion in Lemma 4, we obtain

$$G_3=\frac{1}{2h}\cdot\left[q(x_{i-t+3/2})\int_{x_i}^{x_i+1} z''(s)(x_i+h-s)\,\mathrm{d}s\right.$$
$$\left.-\,q(x_{i-t-1/2})\int_{x_i-1}^{x_i} z''(s)(s-x_i+h)\,\mathrm{d}s\right]+\mathrm{O}(h).$$

Therefore, we have the estimate

$$|G_3| \leqslant Ch\left\{1+\frac{1}{\max(h,\varepsilon)}\exp\left(-\frac{d\cdot x_{i-1}}{\varepsilon}\right)\right\}.$$

By combining (i)–(iii), we find the estimate

$$|L^h(z(x_i)-z_i)| \leqslant Ch\left\{1+\frac{1}{\max(h,\varepsilon)}\exp\left(-\frac{d\cdot x_{i-1}}{\varepsilon}\right)\right\}.$$

Table 2
CCDS, $N=100$, $\varepsilon=10^{-3}$

Mesh point	Exact solution	$\lvert u(x_i)-u_i\rvert$	$\lvert u(x_i)-u_i\rvert$	$\lvert u(x_i)-u_i\rvert$
		$(s=0.0, t=0.0)$	$(s=0.0, t=0.5)$	$(s=0.1, t=0.4)$
$x=0.000$	-0.766646753	0.008474688	0.011252722	0.010696859
$x=0.010$	-0.765463021	0.001499905	0.001292020	0.000733377
$x=0.040$	-0.761653429	0.001471025	0.001269974	0.000721533
$x=0.500$	-0.663966530	0.001021016	0.001175109	0.000735750
$x=0.960$	-0.513864895	0.000583983	0.001315608	0.000935581
$x=1.000$	-0.499000999	0.000547139	0.001333336	0.000957133

Table 3
CCDS, $N=200$, $\varepsilon=10^{-3}$

Mesh point	Exact solution	$\lvert u(x_i)-u_i\rvert$	$\lvert u(x_i)-u_i\rvert$	$\lvert u(x_i)-u_i\rvert$
		$(s=0.0, t=0.0)$	$(s=0.0, t=0.5)$	$(s=0.1, t=0.4)$
$x=0.000$	-0.766646753	0.004131769	0.005525933	0.005246931
$x=0.010$	-0.765463021	0.000891835	0.000501713	0.000222876
$x=0.040$	-0.761653429	0.000872934	0.000495195	0.000221446
$x=0.500$	-0.663966530	0.000592139	0.000503925	0.000284624
$x=0.960$	-0.513864895	0.000350139	0.000597696	0.000408055
$x=1.000$	-0.499000999	0.000331070	0.000607201	0.000419474

On the other hand, a computation gives

$$|B_0^h(z(0)-z_0)| \leqslant Ch \cdot \left\{1+\frac{1-\exp(-\bar{d}\cdot\rho)}{h}\right\}$$

and

$$|B_1^h(z(1)-z_N)| \leqslant Ch.$$

Hence from Lemma 7, we obtain

$$|z(x_i)-z_i| \leqslant Ch.$$

By combining Lemmas 8 and 9, we prove the following theorem.

Theorem 2. *Let $u(x)$ be the solution of SPMBP and u_i be the solution of CCDS, then $|u(x_i)-u_i| \leqslant Ch$ for all $0 \leqslant i \leqslant N$.*

Table 4
CCDS, $N = 400$, $\varepsilon = 10^{-3}$

Mesh point	Exact solution	$\lvert u(x_i) - u_i \rvert$	$\lvert u(x_i) - u_i \rvert$	$\lvert u(x_i) - u_i \rvert$
		$(s = 0.0, t = 0.0)$	$(s = 0.0, t = 0.5)$	$(s = 0.1, t = 0.4)$
$x = 0.000$	-0.766646753	0.002269263	0.002967386	0.002827638
$x = 0.010$	-0.765463021	0.000449018	0.000247256	0.000107954
$x = 0.040$	-0.761653429	0.000436497	0.000247084	0.000110322
$x = 0.500$	-0.663966530	0.000270225	0.000277443	0.000167879
$x = 0.960$	-0.513864895	0.000146214	0.000327335	0.000232601
$x = 1.000$	-0.499000999	0.000136940	0.000331825	0.000238048

Table 5
CCDS, $s = 0.2, t = 0.3, \varepsilon = 10^{-3}$

Mesh point	Exact solution	$\lvert u(x_i) - u_i \rvert$	$\lvert u(x_i) - u_i \rvert$	$\lvert u(x_i) - u_i \rvert$
		$N = 25$	$N = 50$	$N = 100$
$x = 0.000$	-0.766646753	0.043215589	0.021136838	0.010141073
$x = 0.040$	-0.761653429	0.003753642	0.001267810	0.000173163
$x = 0.600$	-0.634985778	0.004199393	0.001523551	0.000345694
$x = 1.000$	-0.499000999	0.004899051	0.001915699	0.000580930

7. Numerical results

In this section, we present two numerical results for solving the following conservative form of SPMBP:

$$\varepsilon[\sqrt{1+x}u'(x)]' + \left[\frac{1}{\sqrt{1+x}}u(x)\right]' = \frac{1}{2\sqrt{1+x}},$$
$$u(0) - 2u'(0) = 1, u(1) + 4u'(1) = 1.$$

The exact solution of the problem is

$$u(x) = \frac{1+x}{1+\varepsilon} + 2K_1\sqrt{1+x} + K_2(1+x)^{-1/\varepsilon},$$

where

$$K_1 = \left[1 - \frac{6}{1+\varepsilon} - \frac{\varepsilon-2}{\varepsilon+1}\cdot 2^{-1/\varepsilon}\right] \Big/ (4\sqrt{2}),$$
$$K_2 = \left(1 + \frac{1}{1+\varepsilon}\right) \Big/ \left(1 + \frac{2}{\varepsilon}\right).$$

This problem is solved numerically using FMFDS and CCDS, which have been presented in Sections 3 and 5, respectively.

Computed maximum pointwise errors between the FMFDS and the exact solution for a variety of values of ε and N are presented in Table 1. From Table 1, it can be seen that these numerical results are in excellent agreement with the exact solution. They also prove ε-uniformly convergence of order $N^{-1}\ln N$.

A comparison of the exact solution and CCDS with different values of t and s is shown in Tables 2–5. From Tables 2–5, it can be seen that the CCDS are in good agreement with the exact solution and CCDS are ε-uniformly convergent schemes.

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